Description and Instructions for the **SPIRULE*** *by Walter R. Evans*

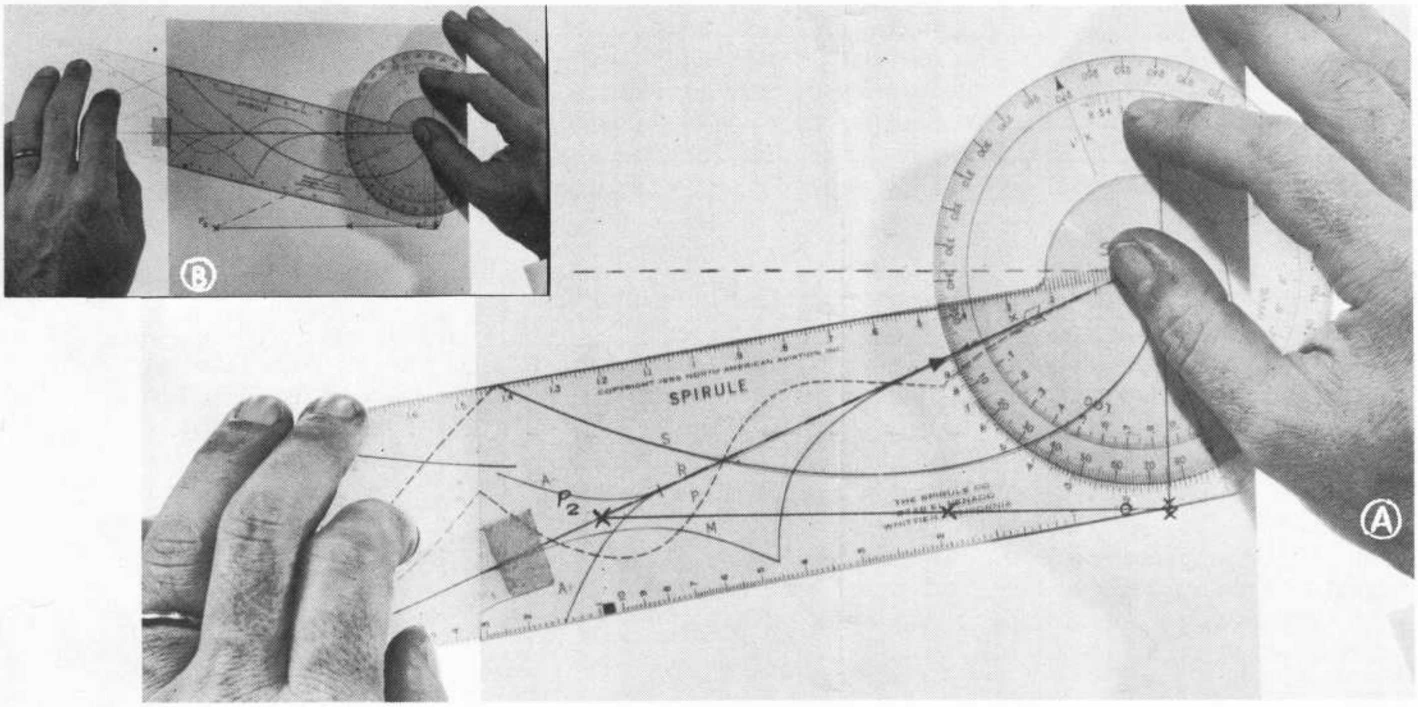


Fig. 1 (a) Spirule in start position: R line of arm at 0° on disk
(b) Spirule after rotation through angle of vector s_1-p_2

The Spirule is based on three ideas: direct measurement on plots, addition of angles by successive rotation of the arm with respect to the disk, and conversion of various quantities to angles by means of curves. The large picture above, Fig. 1(a), shows the Spirule on a root-locus plot. Note that the arrow on the R line of the arm is at 0° on the disk; this is the start position of the Spirule. The center of the disk is at the s_1 point; the line R is alined with the p_2 point. The problem is to add the angles and multiply the lengths of vectors such as s_1-p_2.

Set up the Spirule on Fig. A1 of the insert sheet to follow the sample rotations. Press with the right thumb on the eyelet so that it serves as a fixed pivot. Fix the disk by pressing on it with the right index finger while returning the arm until the line R is horizontal. Note this first angle can be read on the disk at the R arrow as 336°, or -24°. Release the disk and it will turn with the arm while the line R is being alined with the next point p_1. Fixing the disk while rotating the arm until line R is horizontal would add the next angle. Thus addition is achieved by successive rotation of the arm with respect to the disk.

Multiplication in slide rule usage is achieved by addition of logarithms. The curve S is a logarithmic spiral whose angle, with respect to the reference line R, is proportional to the length from the pivot point to the curve. A unit length must have zero angle; this length at the crossing of R and S is 5". A $\log_{10}$ of 1 is an angle of 90°. Set the Spirule at its start position. Note the X1 arrow of the disk is at 1 on the curved scale on the arm at the edge of the disk. With the alinement as in Fig. 1(a) and the pivot fixed as before, fix the disk while turning the arm until the curve S is at the point p_2. Note the reading on the arm as .128 at the X10 arrow of the disk, giving 1.28. This value checks the reading on the scale along the top edge of the arm. Repetition of the rotation for other points adds logarithms and thus multiplies lengths.

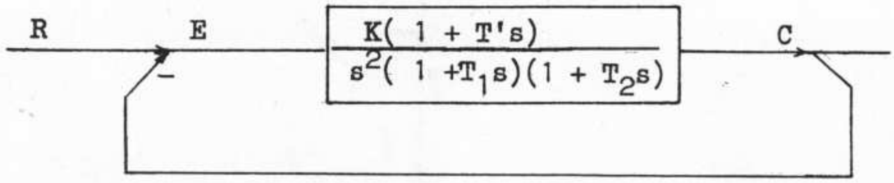
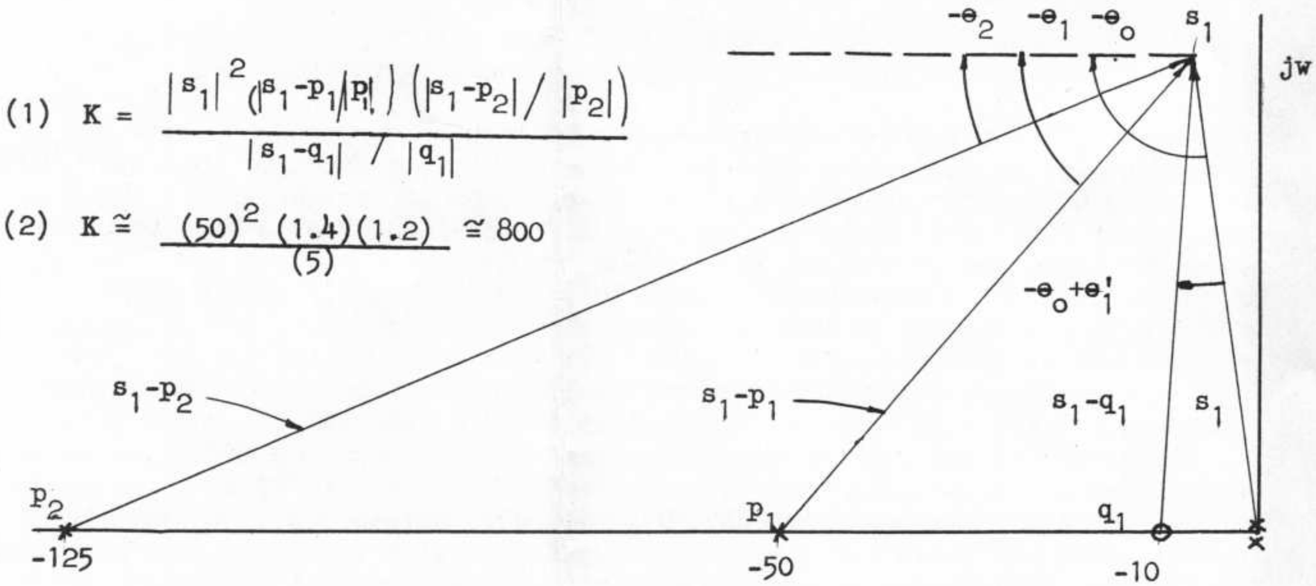


Fig. 2 Block Diagram for System Used for All Plots

Most readers are familiar with block diagrams; others are referred to any text on control systems such as those listed in the References. In root-locus the object is to find the values of the complex number s which make the loop transfer function $C/E = -1$. Trial and error procedure is required: this is simplified graphically by treating a factor $(1+T_1 s)$ as $T_1\ (1/T_1 + s)$. The values, such as $-1/T_1'$, which make a numerator factor zero are called zeros and are marked on the s plane as o's. The values of s, such as $-1/T_1$, which make a denominator factor zero are called poles and are marked on the plot as x's. All factors which depend upon s can then be drawn as vectors from poles and zeros to a common s point as in Fig. 3.



$$(1)\quad K = \frac{|s_1|^2 \left(|s_1-p_1|/|p_1|\right)\left(|s_1-p_2|/|p_2|\right)}{|s_1-q_1| \;/\; |q_1|}$$

$$(2)\quad K \cong \frac{(50)^2\ (1.4)(1.2)}{(5)} \cong 800$$

Fig. 3. Vectors to a Common s Point in a Root-locus Plot

The product of vectors is itself a vector whose angle is the sum of the angles and whose length is the product of the lengths. The requirement that $C/E = -1$ therefore breaks into two conditions: the sum of the angles must be 180°, and the overall magnitude must be 1.

The angle condition can be met by an entire locus of values of s. Special parts can be located first such as intervals along the real axis and asymptotes for very large values of s. Estimating angles permits other parts to be sketched as shown by the dotted line in Fig. 3.

After a particular point s_1 is selected, the magnitude condition can be satisfied by direct calculation of the value of the gain K as in (1) given with Fig. 3.

A ratio of vectors, such as $|s_1-p_1|/|p_1|$, can be readily estimated. The approximate value of K for the given plot is given in (2).

The thought given to sketching the locus and estimating the gain develops an understanding of the effect of zeros and poles. This mental picture is valuable in judging the changes needed to obtain a desired shift in a root. Moreover, after a sketch is made, only the regions of interest need be checked with a Spirule.

VECTOR ANGLE ADDITION (Fig. A2 and Line R)

The object is to check the sum of the vector angles from the point s_1 in Fig. A2. The recommended sequence of steps is as follows:

1) Set the Spirule at its start position with the R arrow at 0° on the disk.
2) Center the pivot eyelet at s_1 and fix it by pressing with the right thumb.
3) Aline R with the pole at the origin, letting the disk turn with the arm.
4) Fix the disk by pressing on it with the right index finger while rotating the arm until the line R is parallel with the negative real axis.
5) Release the disk and aline R with the second pole at the origin.
6) Fix the disk while rotating the R line to the zero q_1. (This step adds the net angle $-\theta_o + \theta_1'$, avoiding motion up to the horizontal and back to q_1.)
7) Repeat steps 3 and 4 for the poles at p_1 and p_2.
8) Read the net angle on the disk at the R line and mark the difference from 180°. (The reading is 180° and the difference 0° in this case.)
9) Try other points, mark the angle differences, and interpolate the locus.

CONVENTIONS AND OPTIONS

The angle scale on the disk increases CCW as is standard for vectors. Lead angles from zeros are considered positive as is standard for direct transfer functions. A net reading of 185° is marked as +5°, meaning an excess of 5° lead. The starting angle can, of course, be 180° as is convenient on every other trial. Rotations for poles decrease the angle reading which can be thought of as decreasing the phase margin. A reading of 355° would be marked -5°.

The option of alining R horizontal and fixing the disk while turning R to the pole can be used if done consistently. It is not recommended because alining R with a pole is the starting point for use of the conversion curves; thus alining R with a pole at the start becomes a convenient habit.

The entire plot can be turned with the negative real axis to the upper left or straight up rather than to the left as pictured. This position allows the plot to be brought closer to the eye without the Spirule arm hitting one's body when the arm is alined with a point near the origin. With the s point nearly below the eye, parallax is reduced; however, this effect is not serious because the lines are only .010" from the bottom surface of the arm.

The thumb is recommended for the pivot eyelet because it is stronger than the index finger and steady pressure is required.

The handle on the arm can be mounted at a shorter radius to reduce arm motion but the left hand tends to block the view of the curves.

SLIDE RULE OPERATION (Fig. A3 and the Curved Scale on the Arm)

Slide rule operation of the Spirule is useful in itself and a helpful step in understanding the spiral curve. The theory is the same as for any slide rule. The numerical scale, for a factor of 10, is located over a 90° arc of the arm. The logarithm itself is marked in one quadrant of the disk. The starting position of the Spirule is now significant in that the X1 arrow of the disk is as 1 on the arm scale and the R line is at 0 on the log scale of the disk.

The steps are given for finding the product (2)(20)(1/5)(60/3). The usual practice is to deal with numbers between 1 and 10 to get the numerical answer and to place the decimal point later. To conform with this practice even though the arm scale is between .1 and 1, the R' line is used instead of R as a reference.

1) Fix the pivot at C and aline R' with the dash D in Fig. A3.

2) To multiply by 2, fix the disk while the arm is turned until .2 on the arm scale is at the X10 arrow of the disk. Note the log reading of .3 on the disk.

3) Release the disk and realine R' at the dash D.

4) To multiply by 2 again, sweep the arm across D to .2 on the scale. This step adds the same angle as in step 2 as shown by the disk reading being .6.

5) To divide by 5, fix the disk while sweeping the arm scale across the dash D from .5 to .1. Note the sense of rotation is opposite from before.

6) To multiply by 6/3, set .3 at D and fix the disk while setting .6 at D.

7) Read the final product as .16 at the X10 arrow, giving 1.6.

The two decimal places in the numerator terms makes the corrected value 160. Note that the usual rough check on the product to see if the value at step 7 is .16 or 16 is not necessary because the only uncertainty with the Spirule is a factor of 10,000 corresponding to a full revolution. In taking the product of many numbers, a typical rotation of the arm would be 45° CCW. A product with three extra numerator factors is thus apt to involve the unmarked arrow, with the factor almost certainly being X100, not X.01.

SPIRAL CURVE (Fig. A3 and Curves S, S/2 and 1/S)

The reasons for the choice of the R line position are: to make the line as long as possible, to be clear of numerical scales, and to allow both senses of rotation to curves. The spiral curve is as long a continuous curve as possible consistent with a convenient unit length. This length is 5" at the intersection of R and S at which the angle and hence the logarithm is zero. The nature of the curve can be checked by the following steps.

1) Set the Spirule at its start position, and fix the pivot at C in Fig. A3. (The scale on the line may not match the Spirule scale due to paper shrinkage.)

2) Aline R with the line on the figure; note that S crosses the scale at 1)

3) Fix the disk and rotate the arm until the S curve crosses the line at .9 on the scale. Note the reading of .9 on the arm scale at the disk.

4) Continue the rotation noting that lengths correspond to arm scale values.

The two steps of measuring a length from pivot to a pole and then setting this number on the curved scale are combined into a single step. The disk is fixed while sweeping the arm across the pole from line R to curve S without reading any number in the process. The curve below .1 is dotted because of the inaccuracy percentagewise involved in measuring short lengths. Beyond 1.4, the actual spiral curve is off the arm. The angle between line R and the missing S curve can be inserted by rotating the arm twice from the line R to the curve S/2. The line 1/S is a reflection of the curve S about the radial edge. If the edge is used as a reference, rotations from the edge to the missing S curve can be inserted by rotating from the dotted 1/S curve to the edge.

COMPUTATION OF GAIN K (Eq. 1, Fig. A2, and the Curve S)

The value of K is given in (1); the vector lengths involved are in Fig. A2. The product of vectors involving s is obtained first; several methods of introducing the other constants and the scale factor correction are given later.

1) Set the Spirule at its start position, and fix the pivot at s_1 in Fig. A2.

2) Aline R with the pole at the origin; then fix the disk while sweeping the arm across the pole from R to the curve S.

3) Repeat step 2 for the second pole at the origin.

4) Aline the curve S at the zero q_1; then fix the disk while sweeping the arm across the zero until the line R is alined with the zero.

5) Repeat step 2 for the poles at p_1 and p_2.

6) Read the net product at the X1 arrow on the disk (.43 in this case)

The actual value on the plot for 5" is 100 rather than 1 as given by the scale for the Spirule. The product must be corrected by the factor $100^{(p-q)}$, in which p is the number of poles and q the number of zeros. The product $|q_1|/|p_1p_2|$ can be expressed from the plot as $10/(50)(125) = 1/625$. The total correction factor is $100^3/625 = 1600$. The value of K is $1600(.43) = 700$.

An alternate method is apt to be more convenient for more involved plots.

1) Set the Spirule at its start position; and fix the pivot at the origin.

2) Sweep across all poles, except those at the origin, from R to S as usual; sweep across all zeros from S to R as is the practice with s as the pivot.

3) The reciprocal of this number is needed; this can be obtained by reading the angle on the disk at the line R and setting this opposite angle. (Read 71° and set -71° or 289° in this case.) An option for this step is to aline R with the negative real axis and mark the 0° mark on the plot; then interchange, putting 0° of the disk at the negative real axis and the R line at the plot mark.

4) The only scale factor correction needed is for the poles at the origin, or 100^2. No setting is needed because 10,000 is a full turn.

5) Note the reading of .16 at the X1 arrow (X10,000 in this case) checks the factor of 1600 obtained above.

6) Mark the disk at the edge of the arm. This mark establishes the starting point for rotations at any s point. The final value is then the gain K.

If the radial edge is used as a reference for the S curve, unity for the Spirule now corresponds to 141 on the plot and here 141 is used for 100 above.

FREQUENCY RESPONSE PLOTS (Fig. 2 and Fig. 3)

Frequency response involves only sinusoidal signals for which s = jw. The transfer function C/E in Fig. 2 can be calculated directly for various values of w. Factors are treated as vectors again; so the sum of the angles and the product of the magnitudes must be found. The s plane plot of zeros and poles can be used in which values of s along the jw axis are used rather than finding a locus by trial and error. The angle addition and calculation of gain are the same as before.

Simplifications are possible because the angle and magnitude of a 1+jwT term are just functions of wT. By plotting curves versus log w, the same curves can be used for any real value of T. Similar curves for quadratic functions in which the zeros or poles are complex have damping ratio as a parameter. The curves for a 1+jwT term are plotted in Fig. 4.

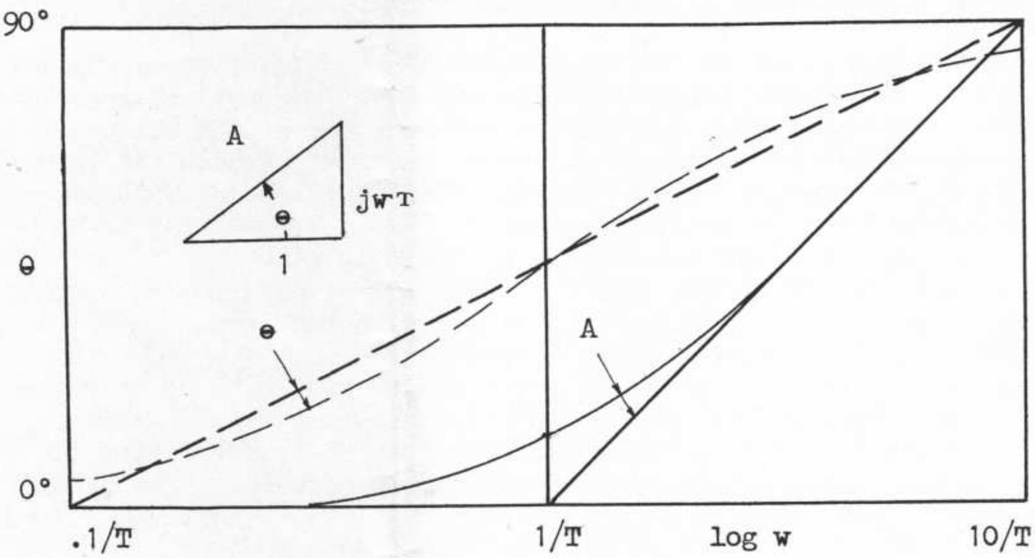


Fig. 4 Log A vs. Log w and θ vs. Log w Curves

The scale for the above plot is $2\frac{1}{2}$" for a factor of 10 in w and $2\frac{1}{2}$" for a factor of 10 in A or 90° in phase angle θ. The approximation for A is 1 for w less than 1/T, and wT for wT greater than 1. The resultant straight lines break at 1/T, which thus gets the name "break frequency". Note the value 1/T is the same as that of a zero or pole on a root-locus plot; for convenience the same symbols o and x are used on the log w axis. The angle curve can be approximated to within 6° by a straight line. The line breaks at .1/T and has a slope of $\frac{1}{2}$ until it breaks level at 10/T.

ADDING ORDINATES (Fig. A4 and Curves A+ and A-)

The simplest method of obtaining the total angle is to trace the standard θ vs. log w curve for each factor and then add ordinates at each value of w. For the given function in Fig. 2, the curves are given in Fig. A4. All curves are plotted upward, but it is understood that ordinates for the poles must be subtracted from those for the zero. The $(jw)^2$ factor gives a constant 180° vs. log w. Phase margin is thus found from the sum of the given ordinates.

The Spirule can add angles; therefore curves are needed on the arm to give angles proportional to ordinates. These curves are marked A+ and A-; their use is explained below for finding the sum of ordinates at w_1 in Fig. A4.

1) Set the Spirule at its start position, aline the R line with the base line of the figure, shifting it to the right until the short vertical line at apex of the A+ and A- curves is at w_1. Note that the pivot is off the paper, and therefore the paper must be prevented from slipping.

2) Fix the disk while turning the arm until the A+ curve is at the ordinate of the q_1 curve. This sense of rotation is not related to conversion curves.

3) Aline the A+ curve at the p_1, ordinate, and fix the disk while turning the arm until R is along the base line. (In this case, the correction effect of the A+ curve is negligible.) Note steps 2 and 3 can be combined by fixing the disk while turning the A+ curve from the p_1 curve to the q_1 curve.

4) Fix the disk while turning the arm from A+ at p_2 until R is level.

5) Aline the X1 arrow horizontal; the net ordinate is then under the Spirule at the intersection of the A+ curve with the w_1 line. Put a pencil at that point, remove the Spirule, and mark the point.

6) If a check is needed, realine the Spirule as in step 5, judge the correction by eye, and mark it accordingly.

Note that the A+ and A- curves can be used for any type of plot; the A- curve is used for ordinates plotted below the reference line of the plot. The base length to the apex of the A+ and A- curves is 5.73"; thus an ordinate of .1" subtends an angle of 1/57.3 rad. or 1°. If a net ordinate cannot be plotted as in step 5, use the radial scale on the arm; a 10° disk reading is 1" or .2 on the scale. The curves are usable to 4.5" or 45°.

STRAIGHT LINE PLOTS (Fig. A5, Curves M and P, Slope Quadrant)

The straight line approximations given in Fig. 4 permit addition to be achieved simply by choice of slopes. At $w = q_1$, the approximate C/E function is simply K/q_1^2. For $q_1 = 10$, the gain must be 100 for C/E to be 1. Below $w = q_1$, the C/E function is K/w^2 which is plotted as a line with a slope of -2. Above $w = q_1$ the $1-s/q_1$ factor is treated as $jw/-q_1$ which changes the slopes of the C/E function from -2 to -1. Similarly, the slope breaks to -2 at p_1 and to -3 at p_3 as shown in Fig. A5. The values of K needed to make C/E = 1 increase as w increases are marked by the scale on the right.

The angle plot is also a series of straight lines as shown in Fig. A5. For very low values of w, the angle is 180° or the margin is 0°. For convenience, mark the zero and pole pattern shifted ahead by a decade and mark them (dotted) back a decade as a reminder of the break points. The first slope break is to 1/2 at $.1q_1$, back to level at $.1p_1$, and so forth.

Slopes are marked on one quadrant of the disk for use in drawing these lines. To draw the line with a slope of 2 for w less than q_1, set the edge of the arm at 2 on the slope scale (near 210°). Put a pencil point at the break point, press either edge of the Spirule against the pencil, and rock the Spirule until the arrows of the disk are square with the coordinates of the plot. Of course, the Spirule can be shifted along its edge to make the arrows fall on plot lines if desired in this alinement.

The straight line plots can be corrected for the approximations used in Fig. 4 by use of the M and P curves. Note that the corrections at a particular frequency, such as w_1 are just a function of the distance on the log w scale between w_1 and the corresponding break frequency.

1) Tape or weight the figure down. Set the Spirule at its start position.

2) Aline R with the top line of Fig. A5, shifting horizontally until the R, S, and P intersection falls at w_1. (This has the pivot 5" or two decades to the right of the frequency being corrected.)

3) Fix the disk while sweeping the arm from curve M at q_1 until curve M is at p_1. (This step combines two operations as often occurs.)

4) Fix the disk while sweeping from R horizontal until curve M is at p_2.

5) Read the net correction (1.03 in this case) and plot a point above the straight line using the $2\frac{1}{2}$" log scale at the bottom far edge of the arm.

The procedure for the angle correction is very similar.

1) Repeat steps 1 and 2 above.

2) Repeat steps 3 and 4 using curve P instead of curve M.

3) Read the correction on the disk at the R line (10° in this case) and plot the point above the given line, using the scale at the end of the arm.

Corrections at p_1 and half way between w_1 and p_1 are most significant because a good gain setting would be about 400 which allows a phase margin of about 30° with C/E = 1. The exact choice depends upon many things.

COMPLETION OF ANALYSES

The operations thus far cover the first row of blocks in Fig. 5 below.

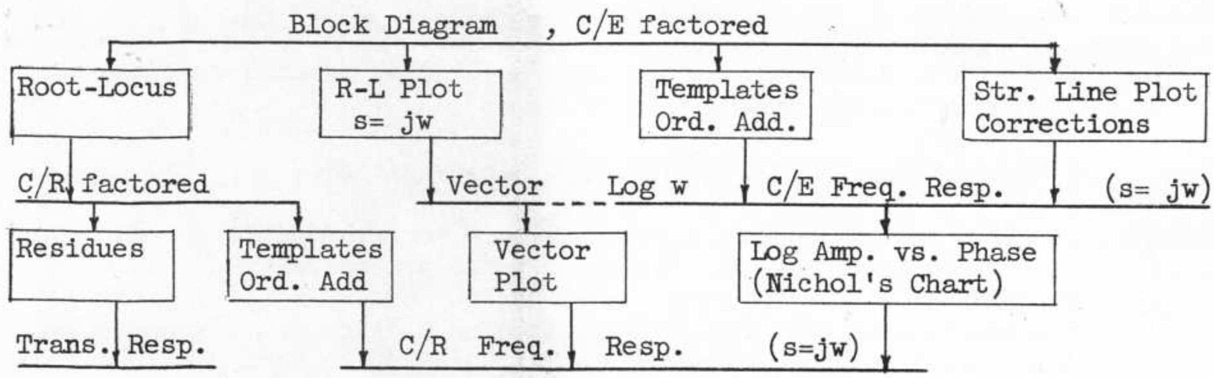


Fig. 5. Alternate Procedures in Closed-loop Analysis

The root-locus plot gives C/R in factored form so that operations can proceed just as for C/E which is given in factored form. Templates can be used for each factor and ordinates added to find C/R for s = jw. Transient response is a sum of exponential terms whose amplitudes are "residues". These residues can be found by vector multiplication as in root-locus plots.

If C/E is found from s= jw values on an s plane, it can be plotted as a vector locus E/C using the disk angle scale and the linear scale on the top edge of the arm. Shifting the pivot point to -1 on the plot adds 1 to all vectors which permits reading R/C = 1 + E/C.

The log w plots given C/E data as angle and log. of amplitude. A vector plot can be constructed to give R/C. A Log. Amp. vs. Phase plot is more convenient, however, in that closed loop values of log. of amplitude and phase can be read directly.

ADDITIONAL USES OF THE SPIRULE

The Spirule arm is $2\frac{1}{2}$" wide which is useful in laying out the $2\frac{1}{2}$" grid for the log plots; curves M and P are based on $2\frac{1}{2}$" per cycle.

The damping ratio scale in one quadrant of the disk is for plotting or reading the damping ratio of complex zeros or poles, in the s plane.

The $2\frac{1}{2}$" and 5" log scales on the bottom edge of the arm are useful on any log plot having those cycle lengths. To add two ordinates, place 5 of the Spirule scale at the top ordinate, read the lower ordinate (e.g. 3.7), and plot the total. (8.7 in this case)

REFERENCES

Bower, John L. and Schulteiss, Peter M: "Introduction to the Design of Servomechanisms", John Wiley & Sons, Inc., New York, 1958

Evans, Walter R.: "Control System Dynamics", McGraw-Hill Book Co. Inc., New York, 1954

Savant, Jr., C.J.: "Basic Feedback Control System Design", McGraw-Hill Book Co. Inc., New York, 1958

Printed in U.S.A.

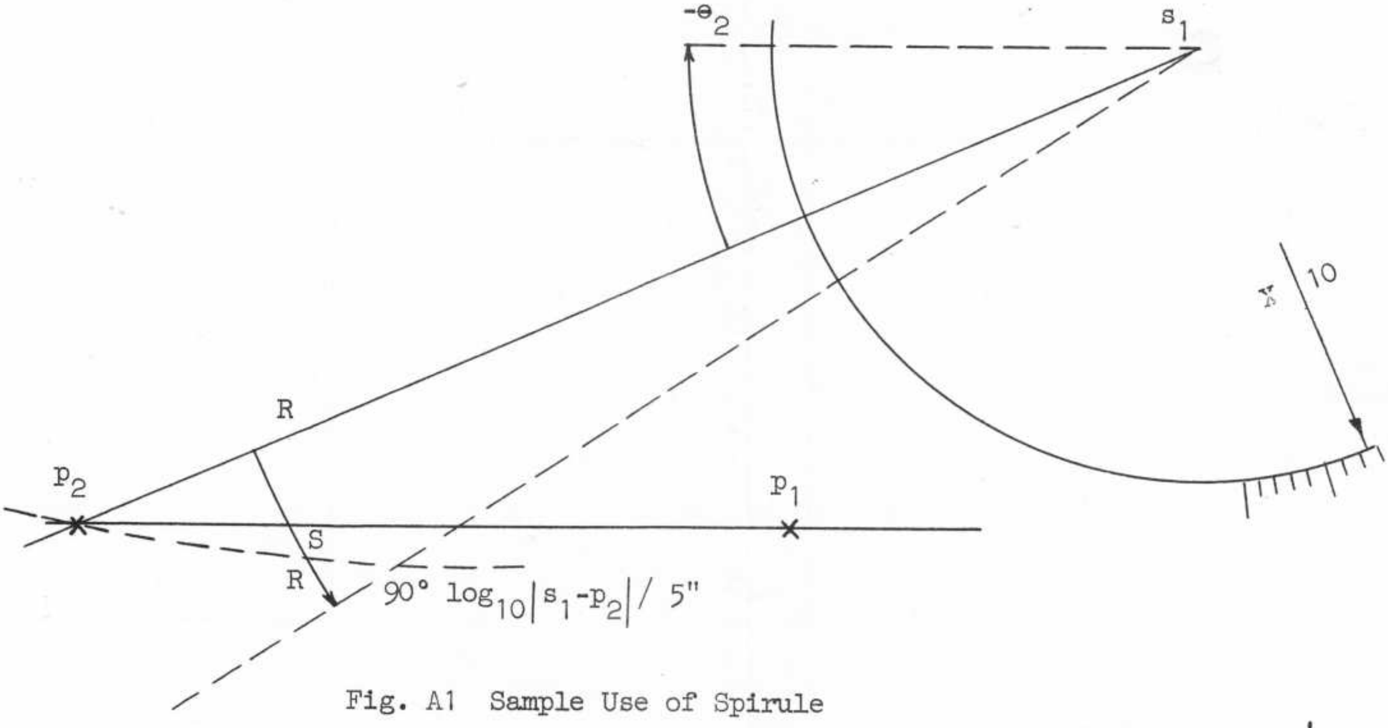


Fig. A1 Sample Use of Spirule

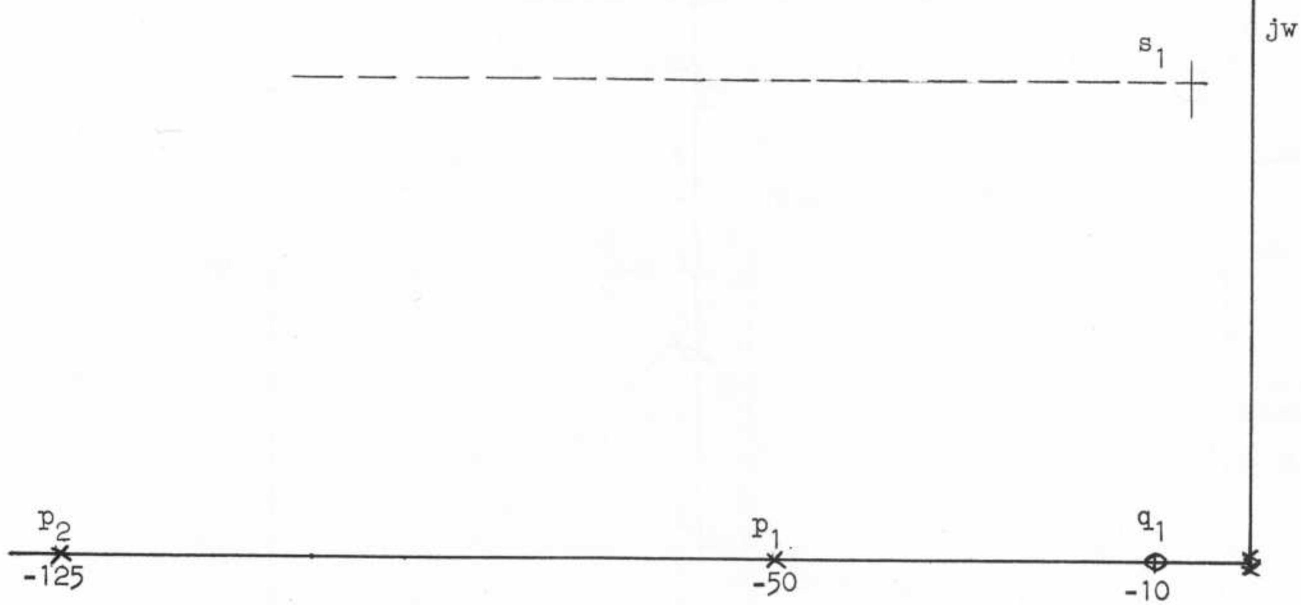


Fig. A2 Zeros and Poles for Root-locus Plot

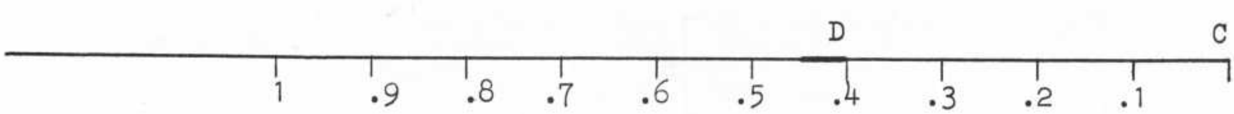


Fig. A3 Slide Rule Operation and Spiral Curve

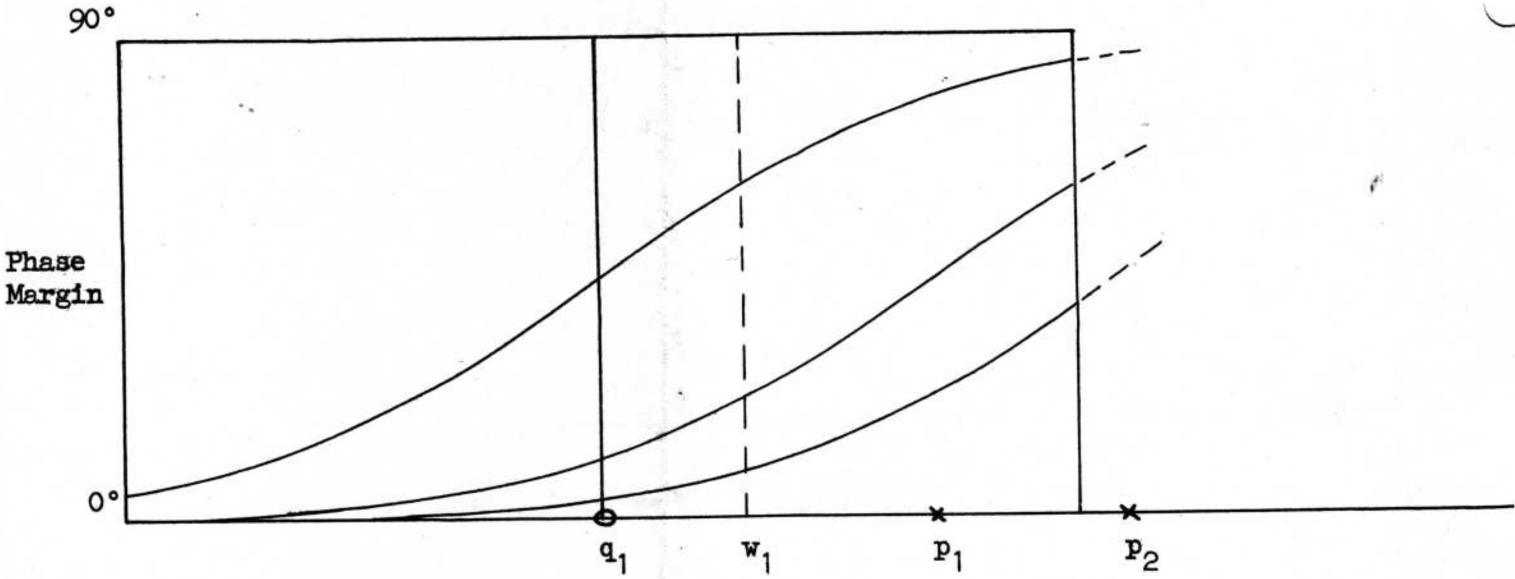


Fig. A4 Addition of Angle Ordinates

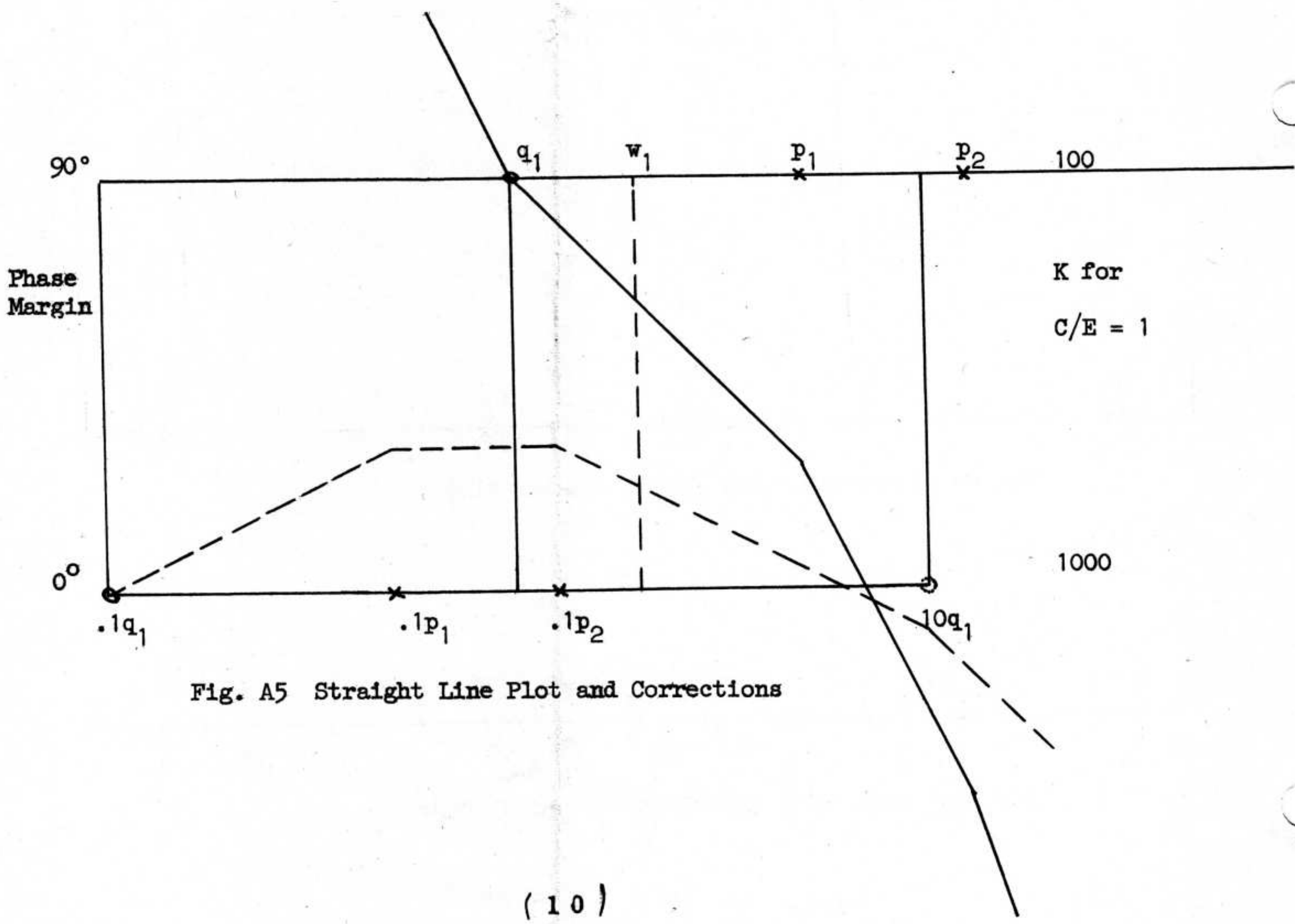


Fig. A5 Straight Line Plot and Corrections